February 12

Talk to Strong Group on Feb 12th

Inspiral tests of general relativity using gravitational waves

Measurements of gravitational waves enable precision tests on relativistic dynamics in highly curved spacetimes. In the first half, I will review the parameterized inspiral tests for binaries with an emphasis on the physical interpretation. I will explain how both the conservative and dissipative sectors of gravity imprint distinct signatures on the gravitational-wave phase, and how these effects appear in the stationary-phase approximation. In the second half, I will discuss parameter estimation in Gaussian noise and how waveform-systematics biases arise, using the Cutler-Vallisneri framework. I will then show how the parameterized tests of general relativity connect to the geometric structure of waveform families in the noise-weighted inner product space and what this implies for identifiability and degeneracies among deviations from general relativity. I will conclude with results of my work testing how robust parameterized inspiral tests are to generic, non-power-law deviations, using the Cutler-Vallisneri bias framework.

Parameterized Tests of Inspiral

Assuming quasicircular inspiral in the stationary phase approximation, the waveform looks like

$$h(f) = A(f)e^{i\Psi(f)} \tag{1}$$

The GR phase looks like

$$\Psi(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3}{128\eta}(\pi M f)^{-5/3}\sum_{k=0}^{7}\phi_i(\pi M f)^{k/3} \tag{2}$$

where $\phi_i = \phi_i(\boldsymbol{\theta}_{\rm int})$. Parameterized tests of GR add extra power-law in frequency deviations to be added [Yunes+ 2009; Agathos+ 2014; Mehta+ 2023]

$$h(f) = A(f)e^{i\Psi(f)+i\Delta\Psi(f)} \tag{3}$$

$$\Delta\Psi_k(f) = \frac{3}{128\eta}\phi_i\delta\phi_k(\pi M f)^{(k-5)/3} \tag{4}$$

Ok, this is cool and all, but why are we looking for deviations of this form? It is motivated by the fact that inspiral GW are moving slowly compared to speed of light. Note that one can show that

$$\frac{M}{r} = \left(\frac{v}{c}\right)^2 = \left(\pi\frac{GM}{c^3}f\right)^{2/3} \tag{5}$$

so in essence, the parameterized tests are counting in powers of $k/2$PN deviations away from GR. A PN power means it is happens with a particular velocity dependence that is different than GR, so order counting in this way seems natural.

All of this is intrinsically related to the stationary phase approximation (SPA). The chirp rate of the signal is related by this identity, encoding both the conservative and dissipative sectors of the binary.

$$\frac{df}{dt} = \underbrace{\frac{df}{dE}}_{\text{conservative}} \times \underbrace{\frac{dE}{dt}}_{\text{dissipative}} \tag{6}$$

- Modifications to Kepler's third law $(\pi M f)^2 = M^3/r^3 + \partial_r \Delta V(r)$ would change how the binding energy relates to frequency (df/dE)
- Extra radiation like scalar dipole radiation or changes to GW radiation would modify luminosity (dE/dt)
- Generally extra radiation, if present is the dominant term in bGR but really depends

Parameterized tests directly relate to the physics in Eq. (6) because you can add fractional changes to the binding energy and luminosity at leading PN, and identify for a particular theory which one is dominant!

SPA shows that the frequency domain phase from Eq. (6) results in a time frequency relation of

$$\Psi(f) = 2\pi \int df t(f) \tag{7}$$

where $t(f)$ is the time frequency relation from SPA, it is the time at which binary is at a particular frequency. One simple result you can show is (SM of [Seymour+ 2026])

$$\Delta\Psi(f) = -2\Delta\phi_{\rm orb}(t_{\rm gr}(f)) \tag{8}$$

which simply relates all of this to the orbital features directly.

Note that different systems measure parameterized coefficients better or worse!

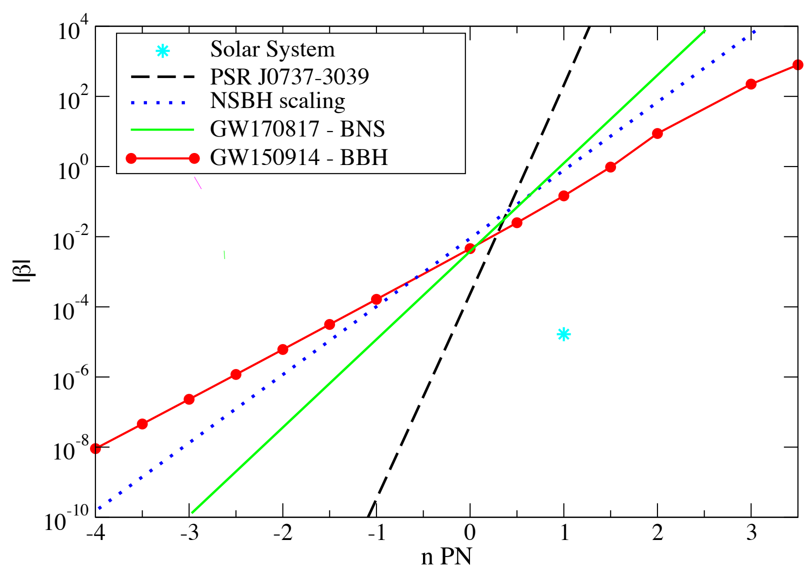

Figure 1: Plot originally from [Yunes+ 2016]. I added approximate curves for BNS and NSBH scaling to illustrate mass dependence on these parameterized tests and cleaned it up.

Gaussian Noise

GW data is a combination of both signal and noise

$$d = s + n \tag{9}$$

where $s = h_{\rm tr}(\boldsymbol{\theta}_{\rm tr})$ and n is gaussian noise. We will describe how Gaussian noise works here

I will review the main method for detection/parameter estimation of GW parameters in gaussian noise (primarily based on [Finn 1992]). Gaussian noise facts

- $\langle n(t) \rangle = 0$
- $\mathrm{cov}\left(n(t), n(t+\tau)\right) = C(\tau)$

Stationarity implies

$$\langle n(f) n^*(f') \rangle = \frac{1}{2}\delta(f-f')S(f) \tag{10}$$

where $S(f)$ is one sided PSD

$$C(\tau) = \int_0^\infty df e^{-2\pi i f \tau} S(f)$$

Note: Eq. (10) is only true for a *particular time stretch*, in practice for LIGO this is only for around *10 minutes until the detector PSD is no longer stationary.*

The result of this is the likelihood is now

$$p(n) \propto \exp\left[-\frac{1}{2}\left(n|n\right)\right] \tag{11}$$

with

$$(a|b) \equiv 4\Re \int_0^\infty df \frac{a(f) b^*(f)}{S(f)} \tag{12}$$

Alright, let's measure parameters. The posterior is

$$p(\boldsymbol{\theta}|d) \propto \exp\left[\frac{1}{2}\left(d - h(\boldsymbol{\theta}) | d - h(\boldsymbol{\theta})\right)\right] \times (\text{priors}) \tag{13}$$

One can show the maximum likelihood estimator is

$$\Gamma_{ij}\left(\theta_j^{\text{MLE}} - \theta_j^{\text{tr}}\right) = (\partial_i h | n) \tag{14}$$

where Γ is fisher information matrix

$$\Gamma_{ij} = (\partial_{\theta^i} h | \partial_{\theta^i} h) \tag{15}$$

Therefore,

- *Average value is true value*: $\langle \theta_i^{\text{MLE}} \rangle = \theta_i^{\text{tr}}$
- *Variance away from mean* is $\langle \Delta\theta_i \Delta\theta_j \rangle = \left(\Gamma^{-1}\right)_{ij}$ where $\Delta\theta_i \equiv \theta_i^{\text{MLE}} - \theta_i^{\text{tr}}$.

This is the fisher matrix technique because square root of diagonal entries are std of how well we can measure it. Thus, the posterior for uniform priors is

$$p(\Delta\theta|d) \propto \exp\left[-\frac{1}{2}\Gamma_{ij}\Delta\theta^i\Delta\theta^j + \mathcal{O}(\Delta\theta^3)\right] \tag{16}$$

Cramer-Rao bound says approximately that this the estimated variance from inverse of Fisher information matrix in Eq. (15) and in Eq. (16) gives the *lower limit on the variance compared to a full Bayesian model.*

- When SNR is very large Cramer-Rao bound gives limiting case of variance.
- Generally, when there is (a) strong degeneracies (b) multimodalities in posterior (c) skewness (ie 'banana'-shaped posterior) and (d) low SNR, the Cramer-Rao bound *underestimates the variance* [Vallisneri 2008].
- If the prior is *more informative* than the likelihood, the Cramer-Rao bound can *overestimate variance* [Rodriguez+ 2013]. This sort of thing can happen for poorly constrained parameters like spin parameters which are between $-1 < \chi < 1$ and also ψ polarization angle if it were a Livingston-Hanford detection[1].

What if instead, our waveform model is incorrect?

$$d = h_{\rm tr}(\boldsymbol{\theta}_{\rm tr}) + n \tag{17}$$

while $h_{\rm rm}(\boldsymbol{\theta})$ is the model. It turns out that the maximum likelihood estimator is

$$\theta_i^{\rm MLE} - \theta_i^{\rm tr} = \underbrace{\left(\Gamma^{-1}\right)_{ij}(\partial_j h_{\rm m}|n)}_{\text{noise}} + \underbrace{\left(\Gamma^{-1}\right)_{ij}(\partial_j h_{\rm m}|h_{\rm tr}(\boldsymbol{\theta}_{\rm tr}) - h_{\rm m}(\boldsymbol{\theta}_{\rm tr}))}_{\text{bias}} \tag{18}$$

This bias is Cutler-Vallisneri bias [Cutler+ 2007].

Geometric description of Cutler-Vallisneri bias

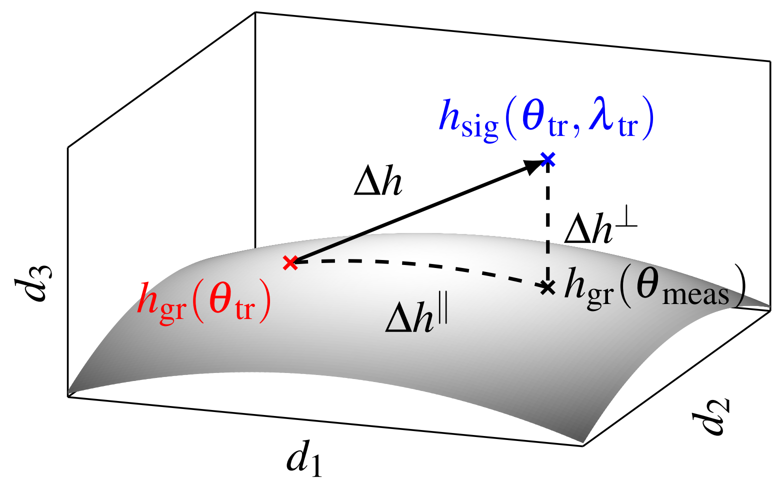

Figure 2: Illustration of GR manifold and a signal that is lying off it. Shows how the residual signal $\Delta h^\perp$ geometrically and how GR parameters get 'stealth bias' without using correct waveform.

If we define $\Delta h \equiv h_{\rm tr}(\boldsymbol{\theta}_{\rm tr}) - h_{\rm m}(\boldsymbol{\theta}_{\rm tr})$, then we related this to the geometric structure as done in [Cutler+ 2007] originally, but specifically in GR in [Vallisneri 2012; Vallisneri+ 2013]. One can show that the GW noise-weighted inner product results in a way to identify 'parallel' and 'perpendicular' deviations from the GR waveform manifold. Specifically, one can show

$$\Delta h_\perp \equiv \Delta h - \Delta\theta^i_{\rm bias}(\partial_i h_{\rm m}|\Delta h) \tag{19}$$

[1] since both detectors are nearly same angle.

- For testing GR, it is not the *nominal deviation* $\|\Delta h\|$ which determines detectability but the *residual deviation* $\rho_\perp \equiv \|\Delta h_\perp\|$!
- $\Delta h_\parallel$ is related to how much GR parameters are biased – see Fig. 2.

Generic tests of GR – geometric description

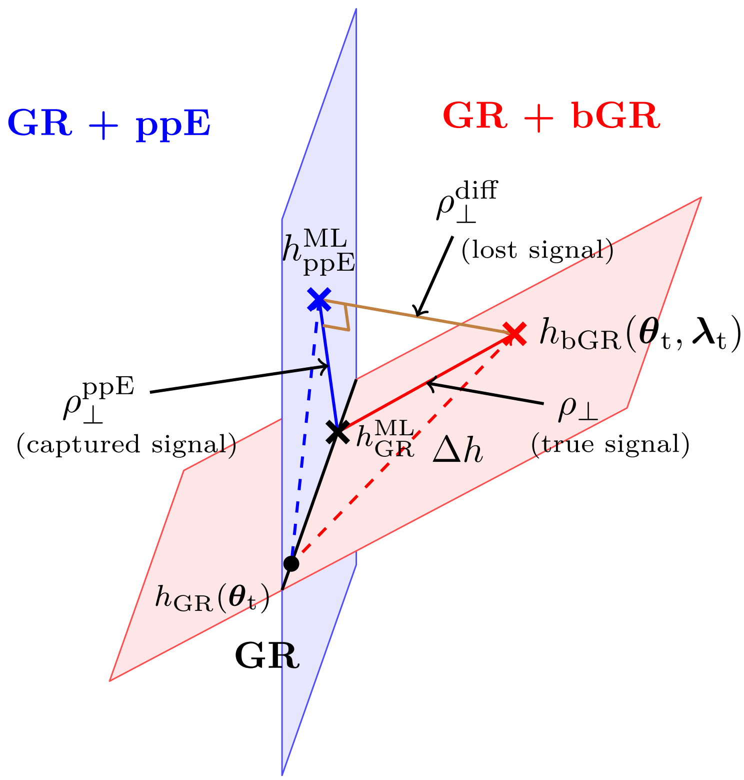

Figure 3: Geometric visualization of how well a parameterized test (blue manifold) can capture true bGR deviations (red manifold), and their intersection is the GR manifold (black).

Suppose instead that there is some true beyond-GR signal

$$h_{\rm bGR}(\boldsymbol{\theta}_{\rm t}, \lambda_{\rm t}) = h_{\rm GR}(\boldsymbol{\theta}_{\rm t}) e^{i\lambda_{\rm t}\psi_\lambda(f)} \tag{20}$$

We are looking for it with a waveform template of a parameterized test using a power law deviation in inspiral.

$$h_{\rm ppE}(\boldsymbol{\theta}, \varphi) = h_{\rm GR}(\boldsymbol{\theta}_{\rm t}) e^{i\varphi_{\rm t}\psi_\varphi(f)} \tag{21}$$

How well can this capture bGR effects? Answer:

$$\rho_\perp^{\rm ppE} = \mathcal{O}(\Delta h^\perp_{\rm ppE}, \Delta h^\perp_{\rm bGR})\rho_\perp \tag{22}$$

where the overlap is

$$\mathcal{O}(\Delta h^\perp_{\rm ppE}, \Delta h^\perp_{\rm bGR}) \equiv \frac{\left(\Delta h^{\perp \rm GR} | \partial_\varphi h^{\perp \rm GR}\right)}{\|\Delta h^{\perp \rm GR}\| \|\partial_\varphi h^{\perp \rm GR}\|} \,. \tag{23}$$

This can be geometrically seen in Fig. 3 with the red and blue manifolds. Note that to distinguish between ppE and bGR models, the thing that matters is

$$\rho_\perp^{\text{diff}} \equiv \sqrt{1-\mathcal{O}^2}\rho_\perp \tag{24}$$

which is also visible in the figure! This can be related to Bayes factors and everything but I will not include this for time.

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